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Classical and quantum Liouville dynamics of SU(1, 1) coherent states

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Abstract. In this paper we contrast the classical and quantum dynamics of SU(1, 1) coherent states by the use of a positive-definite quasiprobability distribution, a Q-function, defined over these states. For Hamiltonians that are linear in the SU(1, 1) generators, therefore coherence preserving, the quantum and classical Liouville equations are identical. This is illustrated for a degenerate parametric amplifier. For an anharmonic oscillator, the quantum Liouville equation contains additional terms containing second-order derivatives with non-positive-definite coefficients. These terms give rise to the quantum recurrences not seen in the classical evolution of this system. It is pointed out that the quantum equation for the Q-function goes over to the classical equation as the Bargmann index, k, becomes large, in agreement with previous semiclassical considerations.

1. Introduction

In recent years there has been much interest in the use of quasiprobability distributions in quantum mechanics and quantum optics [1]. Of particular interest is the contrasting of the quantum and classical dynamics of nonlinear oscillators [2] and rotators [3] by the use of a quantum joint-phase space probability distribution known as the Q-function. The quantum Q-function, unlike the P-function or Wigner's distribution function which may take on negative values, is defined as a true, positive, probability distribution for the simultaneous (approximate) measurement of position and momentum variables. Milburn [2] has studied the quantum and classical Liouville dynamics of a solvable model of an anharmonic oscillator. In that study, the initial state of the system was taken as an ordinary harmonic oscillator coherent state, a state of minimum uncertainty and therefore a most classical quantum state. A classical description of the motion was obtained by solving the corresponding Liouville equation for the classical Q-function. It was shown that an initial Gaussian contour develops a phase space structure called a 'whorl' which becomes more convoluted on a finer scale as $t \to \infty$. This in turn could be associated with the decay of the mean amplitude of the motion in phase space. On the other hand, the exact quantum Q-function does not exhibit the whorls of the classical case but rather shows recurrences without the shear. (However, the contours do become distorted because squeezed states are generated by the interaction.) These quantum recurrences are attributed to the fact that the quantum Q-function satisfies a Liouville-like equation but with additional second-order derivatives with non-positive-definite coefficients. The model anharmonic oscillator produces self-squeezing of the light.

In more recent work, Sanders [3] has studied the classical and quantum dynamics of a nonlinear rotator, again with the use of a Q-function. However, in that case the relevant algebra was that of the SU(2) symmetry group for spin precession rather than the Heisenberg–Weyl group H_4 of the anharmonic oscillator. The Q-function therefore was given as the expectation value of the density operator with respect to coherent states (CSs) associated with SU(2). Similar results to those of the anharmonic oscillator are seen for the nonlinear rotator, namely shearing in the classical Q-function and recurrences in the quantum Q-function.

In the present work we consider another kind of generalized coherent state driven by a nonlinear Hamiltonian. In particular we consider the coherent states of the dynamical group SU(1, 1) driven by a nonlinear oscillator similar to the one discussed in [1]. SU(1, 1) is known to be a dynamical group for a number of systems of general interest in quantum mechanics [4]. CSs over SU(1, 1) have been introduced by Perelemov [5] and have been applied to many of the systems discussed in [4] (see [6] and [7]). On the other hand it has been recognized that the SU(1, 1) CSs generated from a harmonic oscillator ground state are a representation of a particular kind of quantum state known as the squeezed vacuum [8]. Such a state has no classical analogue as its associated P-function in terms of the ordinary or H_4 CSs takes on negative values [9]. It is possible, of course, to introduce a P-function quasiprobability distribution for the SU(1, 1) CSs, as has been done by Wodkiewicz *et al* [10]. In the present paper we work with the SU(1, 1) CS Q-function which is always a positive quasiprobability distribution. In spite of the fact that SU(1, 1) CSs have distinctively non-classical properties, it is still possible to speak of the 'classical' evolution of such states. Hamiltonians that are linear in the generators of SU(1, 1) preserve the coherence of the SU(1, 1) CSs under time evolution, which means that the quantum and 'classical' evolutions are essentially identical [11]. The classical motion in this case takes place in a phase space in the form of the Lobachevsky plane [6, 7] but the corresponding Hamiltonian's equation is nonlinear. It is of course always possible to consider a non-coherence-preserving SU(1, 1) Hamiltonian projected onto this space [12]. However, there is no guarantee that the 'classical' motion in such cases will mimic the true quantum motion [13].

In this paper we study and contrast the evolutions of quantum and classical Q-functions for an initial SU(1, 1) CS interacting with a model anharmonic oscillator of relevance to quantum optics. This model has previously been shown to give rise to self-squeezing of the quantizing electromagnetic field [14] as well as higher-order squeezing [15, 16]. The model Hamiltonian has also been used to describe an optical Kerr medium in one arm of a Mach–Zehnder interferometer [17]. It is related to the Hamiltonian used in [2]. Previously we have studied the interaction of the SU(1, 1) CSs (or squeezed vacuum states) with such a medium and have shown that the squeezing eventually becomes revoked after a short time period [18]. In a more recent study [19] we have compared the quantum and classical time evolution of the SU(1, 1) CSs in anharmonic oscillators with the model to be investigated here as a special case. In that paper the types of motion were compared by calculating the overlap probability of the true quantum state with the 'classical' state which consisted of an undistorted SU(1, 1) CS following the classical orbit. We found that, for *low* excitation of the states, the quantum and classical motions were essentially equivalent but that this was not the case for *high* excitation, somewhat counter-intuitive in regard to the correspondence principle. (This has recently been shown to be the case for the usual H_4 CSs in the same anharmonic oscillator [20].)

This paper is organized as follows. In section 2 we review the essential formalism of SU(1, 1) CSs as will be required for this paper. We also introduce the so-called *P*- and *Q*-representatives of the SU(1, 1) generators which are required for certain statistical averages. In section 3 we introduce the SU(1, 1) *Q*-function and apply it to the case of a coherence-preserving interaction—namely the degenerate parametric amplifier. In this case the quantum and classical *Q*-functions are identical. In section 4 we consider the anharmonic oscillator mentioned above and in section 5 we conclude with some brief remarks. Two appendices have been added to include some mathematical results. The first relates the *P*- and *Q*-representatives of the SU(1, 1) operators while in the second we derive the quantum Liouville equation for the *Q*-function for the anharmonic oscillator.

2. SU(1, 1) coherent states

In this section we briefly review the SU(1, 1) CS formalism that is relevant to the present paper. For a more extensive review, see the cited literature. For a recent review of generalized coherent states see Feng and Gilmore [21] and Kuratsuji *et al* [22].

The SU(1, 1) Lie algebra contains the three operators satisfying the commutation relations

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_-, K_+] = 2K_0 \tag{2.1}$$

along with the Casimir operator

$$C = K_0^2 - \tfrac{1}{2}(K_+K_- + K_-K_+). \tag{2.2}$$

The above algebra may be realized in terms of the boson operators a and a^+ satisfying $[a, a^+] = 1$ as

$$\begin{aligned} K_0 &= \tfrac{1}{4}(a^+a + aa^+) = \tfrac{1}{2}(a^+a + \tfrac{1}{2}) \\ K_+ &= \tfrac{1}{2}a^{+2} \qquad K_- = \tfrac{1}{2}a^2. \end{aligned} \tag{2.3}$$

We require only the positive discrete unitary irreducible representations $\mathscr{D}^+(k)$ where k denotes the Bargmann index such that the eigenvalues of C are $k(k-1)$ and $k>0$. The SU(1, 1) basis states of $\mathscr{D}^+(k)$ are $|m, k\rangle$ such that $K_0|m, k\rangle = (m+k)|m, k\rangle$ where $m = 0, 1, 2, \ldots$. For the realization of the algebra in equations (2.3) one obtains $k = \frac{1}{4}, \frac{3}{4}$. This effectively splits the usual number space into two spaces, one for $k = \frac{1}{4}$ (even photon number) and one for $k = \frac{3}{4}$ (odd photon number).

SU(1, 1) CSs are defined as [5, 6]

$$|\xi, k\rangle = S(z)|0, k\rangle \tag{2.4}$$

where

$$S(z) = \exp(zK_+ - z^*K_-) \tag{2.5}$$

and $z = -(\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi}$, $\xi = -\tanh(\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi}$ where θ and ϕ have the ranges $-\infty < \theta < \infty$ and $0 \leqslant \phi \leqslant 2\pi$. The geometry of the group manifold as parametrized by θ and ϕ actually consist of two unconnected hyperboloids. (This has recently been illustrated in a paper by Aravind [23].) Since it is not possible to jump from one manifold to the other, we henceforth restrict the hyperbolic angle θ to the range $0 \leqslant \theta < \infty$. These coherent states may be expanded as

$$|\xi\rangle = (1-|\xi|^2)^k \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+2k)}{m!\,\Gamma(2k)}\right)^{1/2} \xi^m |m, k\rangle \tag{2.6}$$

where we have dropped the k-label for convenience. These states are not orthogonal since the overlap of two states is given as

$$\langle\xi|\xi'\rangle=(1-|\xi|^2)^k(1-|\xi'|^2)^k(1-\xi^*\xi')^{-2k} \tag{2.7}$$

but they are normalized. Unity may be resolved according to

$$I=\int d\mu(\xi)|\xi\rangle\langle\xi| \tag{2.8}$$

where the measure is given as

$$d\mu(\xi)=\frac{2k-1}{\pi}\frac{d^2\xi}{(1-|\xi|^2)^2}. \tag{2.9}$$

Strictly speaking, these formulae are valid only for $k>\frac{1}{2}$; however, we may always extend k to the region $k\leqslant\frac{1}{2}$ as an analytic continuation of the final results.

We introduce the SU(1, 1) Q-function in the usual way [2] in terms of the density operator $\rho(t)$ as

$$Q(\xi,t)=\mathrm{Tr}(\rho(t)|\xi\rangle\langle\xi|). \tag{2.10}$$

Taking the trace over the SU(1, 1) $\mathscr{D}^+(k)$ basis we have

$$Q(\xi,t)=\sum_m\langle m,k|\rho(t)|\xi\rangle\langle\xi|m,k\rangle=\langle\xi|\rho(t)|\xi\rangle \tag{2.11}$$

where we have used the completeness relation

$$\sum_{m=0}^{\infty}|m,k\rangle\langle m,k|=I. \tag{2.12}$$

It follows from $\mathrm{Tr}(\rho)=1$ and equation (2.11) that

$$\int Q(\xi,t)\,d\mu(\xi)=1 \tag{2.13}$$

as must be the case for a true probability density. Now with

$$\rho(t)=U(t)\rho(0)U^+(t) \tag{2.14}$$

where $U(t)$ is the unitary evolution operator, and taking $\rho(0)$ to be the pure state density operator $\rho(0)=|\xi_0\rangle\langle\xi_0|$ we have

$$Q(\xi,t)=|\langle\xi|U(t)|\xi_0\rangle|^2. \tag{2.15}$$

We now introduce the so-called P- and Q-representatives of operators for the SU(1, 1) CSs. For any operator A defined in the space $\mathscr{D}^+(k)$ it is always possible to write

$$A=\sum_{m,m'}|m,k\rangle A_{mm'}\langle m',k|=\int d\mu(\xi)\,d\mu(\xi')|\xi\rangle\langle\xi|A|\xi'\rangle\langle\xi'| \tag{2.16}$$

where $A_{mm'}=\langle m,k|A|m',k\rangle$. Since the CS $|\xi\rangle$ are overcomplete it is always possible to write the integral kernel of the above in the diagonal form [21]

$$A=\int d\mu(\xi)|\xi\rangle A_P(\xi)\langle\xi| \tag{2.17}$$

where $A_P(\xi)$ is a function defined on the Lobachevsky plane $0 \leq |\xi| < 1$. The function $A_P(\xi)$ is referred to as the P-representative of the operator A and is generally not unique. The Q-representative of the operator A is just the expectation value of A with respect to $|\xi\rangle$, i.e.

$$A_Q(\xi) = \langle\xi|A|\xi\rangle. \tag{2.18}$$

The P-representative of the density operator is

$$\rho(t) = \int \mathrm{d}\mu(\xi)|\xi\rangle P(\xi, t)\langle\xi| \tag{2.19}$$

while the Q-representative has already been given in equation (2.11). Statistical averages in the P-representation are calculated according to

$$\langle A\rangle = \mathrm{Tr}(\rho(t)A) = \int \mathrm{d}\mu \; P(\xi, t)A_Q(\xi) \tag{2.20}$$

while in the Q-representation we have

$$\langle A\rangle = \mathrm{Tr}(\rho(t)A) = \int \mathrm{d}\mu(\xi) \; Q(\xi, t)A_P(\xi). \tag{2.21}$$

Apparently we require the P-representatives of the SU(1, 1) operators in order to calculate their averages with the Q-function. A procedure for finding these functions is given in appendix 1.

Finally, we close this section by making a few remarks on the classical motion of an SU(1, 1) CS. The classical equations of motion for the parameter ξ are [6, 7]

$$\dot{\xi} = \{\xi, \mathcal{H}\} \qquad \dot{\xi}^* = \{\xi^*, \mathcal{H}\} \tag{2.22}$$

where $\{,\}$ is the Poisson bracket defined as

$$\{A, B\} = \frac{(1-|\xi|^2)^2}{2\mathrm{i}k}\left(\frac{\partial A}{\partial \xi}\frac{\partial B}{\partial \xi^*} - \frac{\partial A}{\partial \xi^*}\frac{\partial B}{\partial \xi}\right) \tag{2.23a}$$

$$= \frac{1}{k \sinh\theta}\left(\frac{\partial A}{\partial \phi}\frac{\partial B}{\partial \theta} - \frac{\partial A}{\partial \theta}\frac{\partial B}{\partial \phi}\right) \tag{2.23b}$$

and where

$$\mathcal{H}(\xi, \xi^*, t) = \langle\xi|H(K_0, K_\pm, t)|\xi\rangle. \tag{2.24}$$

In the case of a Hamiltonian nonlinear in the K-operators we use the mean field approximation

$$\mathcal{H}(\xi, \xi^*, t) \simeq H(\langle K_0\rangle, \langle K_\pm\rangle, t). \tag{2.25}$$

We define a classical analogue $Q_{\mathrm{cl}}(\xi, t)$ of the equation Q-function to be the function that satisfies the Liouville equation

$$\frac{\partial Q_{\mathrm{cl}}}{\partial t} = \{\mathcal{H}, Q_{\mathrm{cl}}\} \tag{2.26}$$

subject to some initial distribution $Q_{\mathrm{cl}}(\xi, 0)$. On the other hand, the differential equation satisfied by the quantum Q-function is found by taking the SU(1, 1) CS expectation value of quantum Liouville equation satisfied by the density operator

$$\frac{\partial \rho}{\partial t} = -\mathrm{i}[H, \rho]. \tag{2.27}$$

As we shall illustrate in the next two sections, if H is linear in the SU(1, 1) generators the quantum and classical equations for the Q-function are identical and of first order. For a nonlinear Hamiltonian, the quantum equation will contain additional terms of second order with non-positive coefficients just as for the usual H_4 CSs.

3. Degenerate parametric amplifier

The degenerate parametric amplifier (DPA) has received much attention with regard to the fact that SU(1, 1) is its dynamical group [8, 11, 18, 24]. Such a system is expected to generate squeezed as well as antibunched photon states. Before discussing the model let us review the condition for which a state is squeezed. In what follows we work mainly in the interaction picture.

We define the quadrature operators as

$$X_1 = \tfrac{1}{2}(a + a^+) \qquad X_2 = \frac{1}{2\mathrm{i}}(a - a^+) \tag{3.1}$$

such that

$$[X_1, X_2] = \frac{\mathrm{i}}{2} \tag{3.2}$$

from which it follows that

$$\langle(\Delta X_1)^2\rangle\langle(\Delta X_2)^2\rangle \geqslant \tfrac{1}{16}. \tag{3.3}$$

Squeezing exists if either variance in equation (3.3) is less than $\frac{1}{4}$. Since for an SU(1, 1) state of definite $k(=\frac{1}{4}$ or $\frac{3}{4})\langle X_{1,2}\rangle = 0$, then we have

$$\langle(\Delta X_{1,2})^2\rangle = \langle K_0\rangle \pm \tfrac{1}{2}\langle K_+ + K_-\rangle \tag{3.4}$$

where we have used the realization of equation (2.3).

In the interaction picture, we take the Hamiltonian of the DPA as

$$H_\mathrm{I} = \mathrm{i}\gamma(K_- - K_+). \tag{3.5}$$

The expectation value of H_I is

$$\mathscr{H}_\mathrm{I}(\xi, \xi^*) = \mathrm{i}\gamma(\langle K_-\rangle - \langle K_-\rangle)$$

where, from appendix 1,

$$\langle K_-\rangle = \langle K_+\rangle^* = \frac{2k\xi}{(1-|\xi|^2)} = -k \sinh\theta\, \mathrm{e}^{-\mathrm{i}\phi}. \tag{3.6}$$

Thus from equations (2.23) and (2.26) we have

$$\frac{\partial Q_\mathrm{cl}}{\partial t} = \frac{(1-|\xi|^2)^2}{2\mathrm{i}k}\left(\frac{\partial \mathscr{H}_\mathrm{I}}{\partial \xi}\frac{\partial Q_\mathrm{cl}}{\partial \xi^*} - \frac{\partial \mathscr{H}_\mathrm{I}}{\partial \xi^*}\frac{\partial Q_\mathrm{cl}}{\partial \xi}\right) = \gamma\left((1-\xi^{*2})\frac{\partial Q_\mathrm{cl}}{\partial \xi^*} + (1-\xi^2)\frac{\partial Q_\mathrm{cl}}{\partial \xi}\right). \tag{3.7}$$

To obtain the differential equation satisfied by the quantum Q-function, we take the expectation value of

$$\frac{\partial \rho}{\partial t} = \gamma[(K_- - K_+), \rho]. \tag{3.8}$$

Thus we have, from equations (A2.12) and (A2.16),

$$\frac{\partial Q}{\partial t} = \gamma\left((1-\xi^{*2})\frac{\partial Q}{\partial \xi^*} + (1-\xi^2)\frac{\partial Q}{\partial \xi}\right) \tag{3.9}$$

which is identical to equation (3.7) as expected.

For an initial Q-function given by

$$Q(\xi, 0) = |\langle \xi|\xi_0\rangle|^2 = (1-|\xi|^2)^{2k}(1-|\xi_0|^2)^{2k}(1-\xi^*\xi_0)^{-2k}(1-\xi\xi_0^*)^{-2k} \tag{3.10}$$

then at time t, from equation (2.15),

$$Q(\xi, t) = |\langle \xi|\, e^{-iH_I t}|\xi_0\rangle|^2 = |\langle \xi|\, e^{-2i\gamma t K_2}|\xi_0\rangle|^2 \tag{3.11}$$

where $K_2 = (1/2i)(K_+ - K_-)$. Since it can be shown that

$$e^{-2i\gamma t K_2}|\xi_0\rangle = e^{i\Phi}|\xi_0'\rangle \tag{3.12}$$

where $\Phi = 2k(\cosh \gamma t - \sinh \gamma t \xi_0)$,

$$\xi_0' = \frac{\xi_0 \cosh \gamma t - \sinh \gamma t}{-\xi_0 \sinh \gamma t + \cosh \gamma t}. \tag{3.13}$$

then

$$Q(\xi, t) = (1-|\xi|^2)^{2k}(1-|\xi_0'|^2)^{2k}(1-\xi^*\xi_0')^{-2k}(1-\xi\xi_0'^*)^{-2k} \tag{3.14}$$

where ξ_0' is given by equation (3.13). The contours of the Q-function of equation (3.10) consist of non-concentric circles as seen in figure 1 for $\xi_0 = 0.5$. For $\xi_0 = 0$, the contours would be concentric about $\xi = 0$. For this case (the vacuum if $k = \frac{1}{4}$) we have from equation (3.13) that as $t \to \infty$, $\xi_0' \to -1$ so that the contours would still be circles but the centres move toward $\xi = -1$ as $t \to \infty$. Contours at greater altitude are centred closer to $\xi = -1$ (figure 2). This situation corresponds to increased squeezing in the X_1 quadrature, actually $\langle(\Delta X_1)^2\rangle \to 0$ as $t \to \infty$ [18].

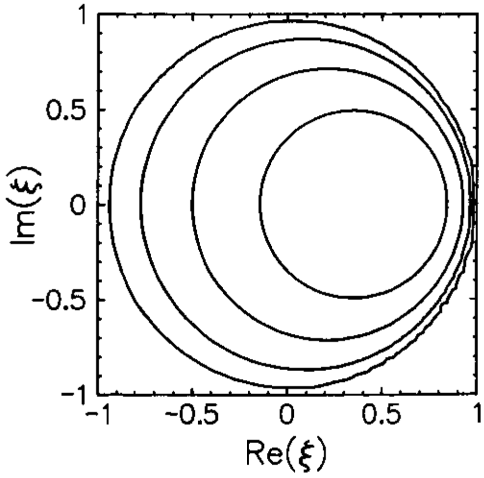

Figure 1. Contour plot of the Q-function of equation (3.10) with $\xi_0 = 0.5$. The contours are separated by 0.2, the central one being at 0.8, the outermost at 0.2. The same spacing is used in all the multicontour plots in this paper.

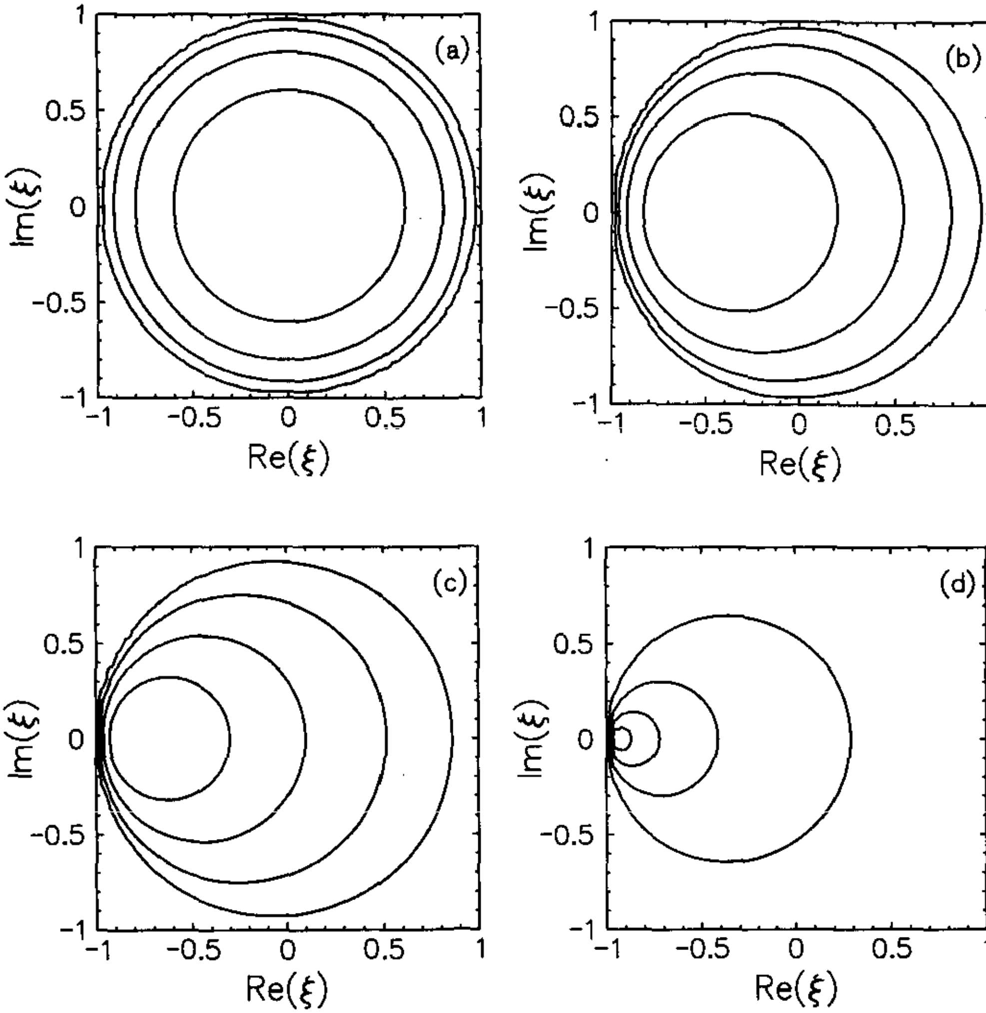

Figure 2. Sequence showing the evolution of a Q-function, with $\xi_0 = 0$, driven by a degenerate parametric amplifier as given in equation (3.14).

4. Anharmonic oscillator

In terms of the annihilation and creation operators of a single-mode field, the Hamiltonian of the anharmonic oscillator describing a nonlinear non-absorbing medium is [14]

$$H = \omega(a^+a + \tfrac{1}{2}) + \frac{\lambda}{2} a^{+2}a^2. \tag{4.1}$$

In terms of the realization of the SU(1, 1) operators of equation (2.3)

$$H = 2\omega K_0 + 2\lambda K_+K_-. \tag{4.2}$$

Since the first term (the free Hamiltonian) commutes with the second then, in the interaction picture, the Hamiltonian is $H_1 = 2\lambda K_+K_-$. As we have said above, this Hamiltonian has already been studied in regard to its effect on the evolution of an initial SU(1, 1) cs [18, 19].

First we consider the classical motion. The classical interaction Hamiltonian in the mean field approximation is

$$\mathcal{H}_1 = \langle\xi|2\lambda K_+K_-|\xi\rangle \approx 2\lambda\langle K_+\rangle\langle K_-\rangle = 2\lambda\frac{4k^2|\xi|^2}{(1-|\xi|^2)^2} = 2\lambda k^2 \sinh\theta^2. \tag{4.3}$$

The equation of motion for θ and ϕ are

$$\dot{\theta} = \{\theta, \mathcal{H}_1\} = 0 \tag{4.4}$$

$$\dot{\phi} = \{\phi, \mathcal{H}_1\} = 4\lambda k \cosh\theta. \tag{4.5}$$

Thus θ is a constant of the motion and

$$\phi(t) = \phi(0) + 4\lambda kt\cosh\theta \tag{4.6}$$

and therefore

$$\xi(t) = \xi(0)\,\mathrm{e}^{-4\mathrm{i}\lambda kt\cosh\theta(0)} \tag{4.7}$$

where

$$\xi(0) = -\tanh\left(\frac{\theta(0)}{2}\right)\mathrm{e}^{-\mathrm{i}\phi(0)}. \tag{4.8}$$

Obviously $|\xi|^2$ is a constant of the motion. Thus the classical motion in the ξ-plane is circular with angular frequency $\Omega = 4\lambda k\cosh\theta(0)$.

From equation (2.6) the equation of motion for the classical Q-function is

$$\frac{\partial Q_{\mathrm{cl}}}{\partial t} = +4\mathrm{i}\lambda k\left(\frac{1+|\xi|^2}{1-|\xi|^2}\right)\left(\xi^*\frac{\partial Q_{\mathrm{cl}}}{\partial\xi^*} - \xi\frac{\partial Q_{\mathrm{cl}}}{\partial\xi}\right) \tag{4.9}$$

or in terms of θ, ϕ

$$\frac{\partial Q_{\mathrm{cl}}}{\partial t} = 4\lambda k\cosh\theta\frac{\partial Q}{\partial\phi}. \tag{4.10}$$

Since no derivatives with respect to θ appear, θ may be treated as a constant. Let $\tau = 4\lambda k\cosh\theta$; we then have

$$\frac{\partial Q_{\mathrm{cl}}}{\partial\tau} = \frac{\partial Q_{\mathrm{cl}}}{\partial\phi}. \tag{4.11}$$

If at $t=0$, Q_{cl} has the form given by equation (3.10), then equation (4.11) implies that the solution has the form of equation (3.10) but with ξ replaced by $\mathrm{e}^{-\mathrm{i}\tau}\xi = \mathrm{e}^{-4\mathrm{i}\lambda kt\cosh\theta}\xi$. Therefore

$$Q_{\mathrm{cl}}(\xi, t) = (1-|\xi|^2)^{2k}(1-|\xi_0|^2)^{2k}(1-\xi^*\xi_0\,\mathrm{e}^{4\mathrm{i}\lambda tk\cosh\theta})^{-2k}(1-\xi\xi_0^*\,\mathrm{e}^{-4\mathrm{i}\lambda tk\cosh\theta})^{-2k} \tag{4.12}$$

where it should be noted that $\cosh\theta = (1+|\xi|^2)/(1-|\xi|^2)$.

In figure 3 we illustrate the time evolution of this distribution. It is apparent that the contours develop 'whorls' just as in the case of the ordinary CSs [2], becoming more convoluted on a finer scale as $t\to\infty$.

We expect this behaviour to be reflected in the moments. However, it is not convenient to average over $\xi(t)$ since, unlike the case of the usual CSs, $\xi(t)$ is not an eigenvalue of an operator. However, the operator K_- is analogous to the annihilation operator. Using the 'classical' SU(1, 1) CS for the anharmonic oscillator

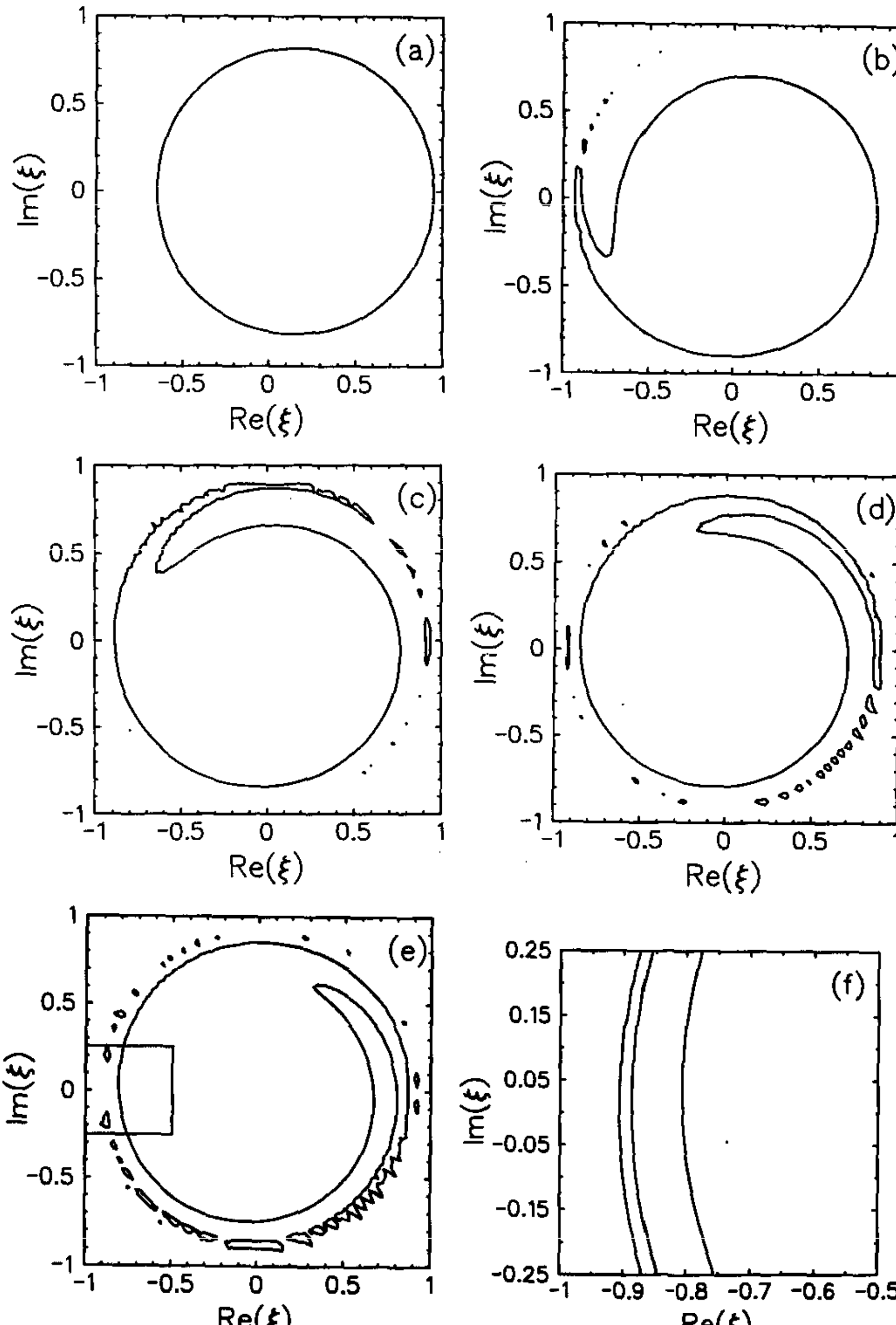

Figure 3. Evolution of a single contour at 0.5 for the classical Q-function of the anharmonic oscillator of equation (4.12). The apparent jaggedness and breakup of the contour is merely an artifact of the grid size.

$|\xi(0)\,\mathrm{e}^{-4\mathrm{i}\lambda kt\cosh\theta}\rangle$, the Q-representative of K_- is from equation (A1.6), taking the complex conjugate

$$(K_{-Q}(t))_{\mathrm{cl}}=\frac{2k\xi(0)\,\mathrm{e}^{-4\mathrm{i}\lambda tk\cosh\theta(0)}}{(1-|\xi(0)|^2)}. \tag{4.13}$$

The real and imaginary parts of this form a representation of phase space. The corresponding classical P-representative is

$$K_{-P}(\xi)_{\mathrm{cl}}=\frac{2(k-1)\xi(0)\,\mathrm{e}^{-4\mathrm{i}\lambda tk\cosh\theta(0)}}{(1-|\xi(0)|^2)} \tag{4.14}$$

and the statistical (ensemble) average is, from equation (2.2),

$$\langle K_-(t)\rangle_{\mathrm{cl}}=\int \mathrm{d}\mu(\xi)\;K_{-P}(\xi)_{\mathrm{cl}}Q_{\mathrm{cl}}(\xi,t). \tag{4.15}$$

Unfortunately it has not been possible to obtain a closed-form expression for this quantity and therefore we cannot make the analytic continuation to $k=\frac{1}{4}$. Instead we break $\langle K_-(t)\rangle_{\mathrm{cl}}$ into its real and imaginary parts and write

$$\langle K_-(t)\rangle_{\mathrm{cl}}=\frac{(2k-1)2(k-1)}{\pi}\left(\iint \mathrm{d}x\,\mathrm{d}y\frac{x}{(1-x^2-y^2)^3}\,Q_{\mathrm{cl}}(x,y,t)\right.$$
$$\left.+\mathrm{i}\iint \mathrm{d}x\,\mathrm{d}y\frac{y}{(1-x^2-y^2)^3}\,Q_{\mathrm{cl}}(x,y,t)\right) \tag{4.16}$$

where $x=\mathrm{Re}\,\xi$, $y=\mathrm{Im}\,\xi$,

$$Q_{\mathrm{cl}}(x,y,t)=(1-x_0)^{2k}(1-x^2-y^2)^{2k}$$
$$\times\left\{1-2x_0x\cos\left[4\lambda tk\left(\frac{1+x^2+y^2}{1-x^2-y^2}\right)\right]\right.$$
$$\left.+2x_0y\sin\left[4\lambda tk\left(\frac{1+x^2+y^2}{1-x^2-y^2}\right)\right]-x_0^2(x^2+y^2)\right\}^{-2k} \tag{4.17}$$

and where we have assumed that ξ_0x_0 (real). The integration is over the unit circle $x^2+y^2<1$. In figure 4, we plot the real and imaginary parts of $\langle K_-(t)\rangle_{\mathrm{cl}}$ for the case

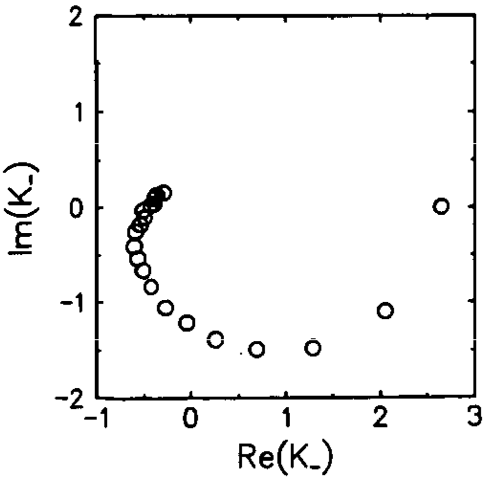

Figure 4. Plot of imaginary against real parts of $\langle K_-(t)\rangle_{\mathrm{cl}}$, with $\xi_0=0.5$, from the Monte Carlo integrating of equation (4.16) with $k=2$.

when $\xi_0=0.5$ and $k=2$. The fact that the curve is not smooth is due to the nature of the Monte Carlo calculation. Nevertheless, we clearly see the motion 'damping' to zero as $t\to\infty$. No true damping is taking place, the approach to zero being a statistical result. We extrapolate from this similar behaviour for $k=\frac{1}{4}$.

Finally we turn to the quantum case. The equation of motion for the density operator is

$$\frac{\partial\rho}{\partial t}=-\mathrm{i}2\lambda[K_+K_-,\rho] \tag{4.18}$$

taking the expectation value of this equation, and using results from appendix 2, we obtain

$$\frac{\partial Q}{\partial t}=-4\mathrm{i}\lambda k\left(\frac{1+|\xi|^2}{1-|\xi|^2}\right)\left(\xi^*\frac{\partial Q}{\partial\xi^*}-\xi\frac{\partial Q}{\partial\xi}\right)-2\mathrm{i}\lambda\left(\xi^{*2}\frac{\partial^2 Q}{\partial\xi^{*2}}-\xi^2\frac{\partial^2 Q}{\partial\xi^2}\right) \tag{4.19}$$

which contains second-order derivatives with non-positive definite coefficients. Note that if k is large we may drop the second-order terms and recover equation (4.9). This is another illustration that the large k limit is the classical limit for SU(1, 1) CSs [25].

The solution to equation (4.19) is actually easier to obtain using the definition of equation (2.15) with

$$U_1(t)=\mathrm{e}^{-2\mathrm{i}\lambda t K_+K_-}. \tag{4.20}$$

Using the fact that

$$K_+K_-|m,k\rangle=[m(m+2k-1)]|m,k\rangle \tag{4.21}$$

we have

$$Q(\xi,t)=[(1-|\xi|^2)(1-|\xi_0|^2)]^{2k}\left|\sum_{m=0}^{\infty}\left(\frac{\Gamma(m+2k)}{m!\Gamma(2k)}\right)(\xi^*\xi_0)^m\,\mathrm{e}^{-2\mathrm{i}\lambda t m(m+2k-1)}\right|^2. \tag{4.22}$$

This solution can also be obtained by using a procedure similar to that used in [2].

We note in passing that it has previously been shown [26] that an initial squeezed vacuum state $|\xi,\frac{1}{4}\rangle$ interacting with an anharmonic oscillator with interaction Hamiltonian of the form $(a^+a)^2$ evolves into a superposition of vacuum states $|\xi,\frac{1}{4}\rangle$ and $|-\xi,\frac{1}{4}\rangle$. This superposition does not arise in the present case.

Let us now consider some quantum mechanical moments. One can use either equation (2.21) or, simply,

$$\langle A\rangle=\mathrm{Tr}(\rho A). \tag{4.23}$$

For the operator K_- we obtain

$$\langle K_-\rangle(t)=\frac{2k\xi_0\,\mathrm{e}^{-4\mathrm{i}\lambda k t}}{(1-|\xi_0|^2\,\mathrm{e}^{-4\mathrm{i}\lambda t})^{2k+1}}(1-|\xi_0|^2)^{2k}. \tag{4.24}$$

The behaviour of this function is illustrated in figure 5. While for short time and large $|\xi_0|$ the quantum and classical motions agree, the quantum mechanical motion shows a complete recurrence, aside from a phase, at time $t=2\pi/4\lambda$. The average of K_0 yields

$$\langle K_0\rangle=k\left(\frac{1+|\xi_0|^2}{1-|\xi_0|^2}\right) \tag{4.25}$$

which is to be expected since K_0 is a constant of the motion. We note, however, from equations (3.4), (4.24) and (4.25), that we recover the previously calculated variances of the field quadratives for the case when $k=\frac{1}{4}$ (see figure 6).

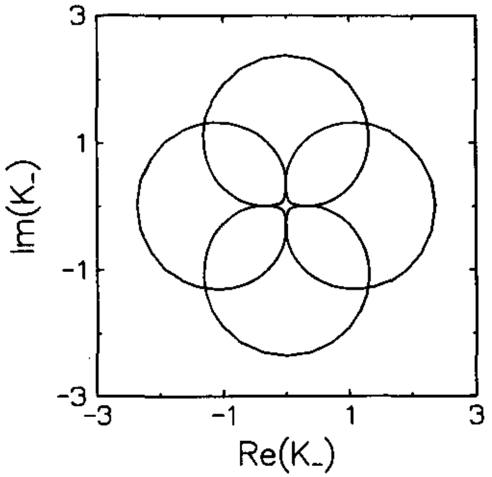

Figure 5. Plot of imaginary against real parts of $\langle K_-(t)\rangle$ from equation (4.24) with $\xi_0 = 0.5$ and $k = \frac{1}{4}$.

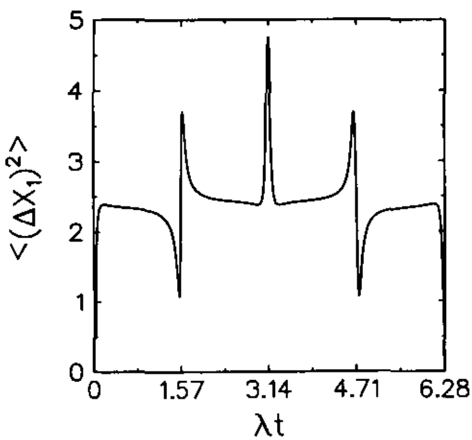

Figure 6. Variance against λt of the X_1 quadrature of the field for $\xi_0 = -0.9$.

Finally, we consider the dynamics of the quantum Q-function itself. This is illustrated in figure 7. It is apparent that the evolution of this function is quite different than for the classical case. We notice that the contours do distort somewhat but also their 'centres' seem to rotate about the origin. This behaviour may be contrasted with the behaviour of the time evolution of the variances in figure 6.

5. Conclusions

In this paper we have compared the classical and quantum evolutions of SU(1, 1) CSs by the use of a Q-function probability distribution that is always positive. Even though the SU(1, 1) CSs contain an inherent quantum property (i.e. squeezing) it is nevertheless possible to obtain a classical picture of these states in terms of their motion on the Lobachevsky plane. For coherence-preserving Hamiltonians, the quantum and classical Liouville equations agree as expected and contain first-order derivatives of Q. (It should be noted that for the DPA with ordinary coherent states, the quantum Liouville equation would contain second- as well as first-order derivatives, an indication that such Hamiltonians do not preserve the coherence of those states.) For the anharmonic

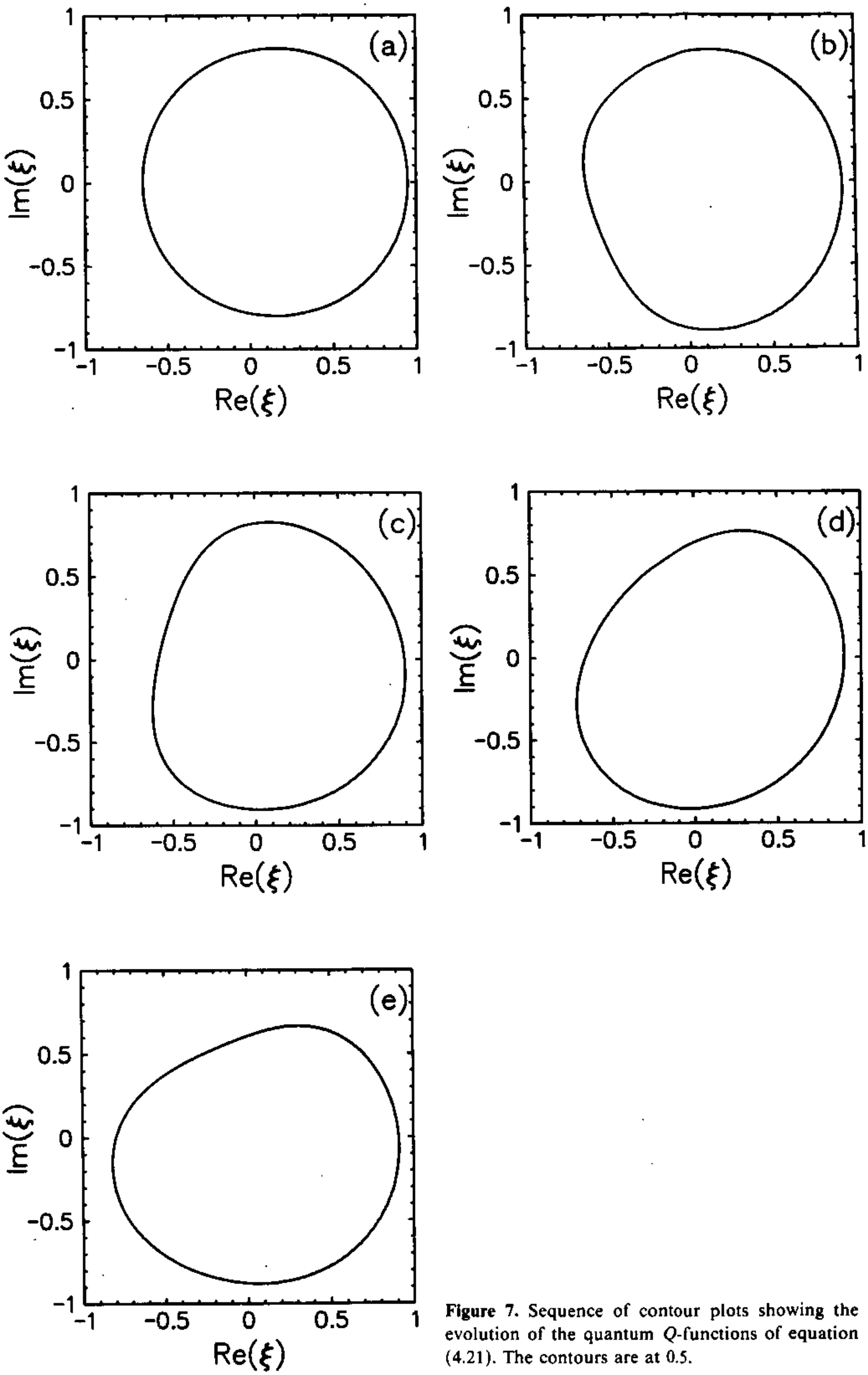

Figure 7. Sequence of contour plots showing the evolution of the quantum Q-functions of equation (4.21). The contours are at 0.5.

oscillator the classical and quantum Liouville equations do differ by the presence of the second-order derivatives in the latter. It is these non-positive-definite diffusion terms that give rise to the observed quantum recurrences, just as in the case of the oscillator CSs. The classical limit for the SU(1, 1) CSs is quite clearly shown to be the limit when k becomes large. In previous work, using a different method [19], we showed that the quantum and classical evolution of the SU(1, 1) CSs in the anharmonic oscillator agreed well for fixed k and low excitation, but that for high excitation the two evolutions are not equivalent, again for the same k. This result is not in disagreement with the results of this paper since our classical results are large-k results arising from neglecting the second-order derivatives in equation (4.19). Elsewhere [25] we have shown that the large-k approximation gives rise to a Bohr–Sommerfeld quantization rule which yields respectable results for the energy eigenvalues of even-powered anharmonic oscillators. In this sense then, the 'classical' results of the present paper might be interpreted as semi-classical.

One immediate use of the SU(1, 1) CS Q-function is in the comparison of the classical and quantum behaviours of a system whose classical counterpart is chaotic [27, 28], where the initial state is an SU(1, 1) CS. Such a study is currently in progress and will be reported elsewhere.

Appendix 1

In this appendix we derive a relationship between the P- and Q-representatives of operators for SU(1, 1) CSs.

For any operator A defined in the space of $\mathscr{D}^+(k)$ the Q-representative is defined as the expectation value with respect to the coherent state

$$A_Q(\xi)=\langle\xi|A|\xi\rangle. \tag{A1.1}$$

If Λ is an exponential operator of the form

$$\Lambda=\mathrm{e}^{\gamma_+K_+}\,\mathrm{e}^{\mathrm{i}\gamma_0K_0}\,\mathrm{e}^{\gamma_-K_-} \tag{A1.2}$$

then by using the non-unitary 2×2 representation of SU(1, 1) we obtain [22]

$$\langle\xi|\Lambda|\xi\rangle=(1-|\xi|^2)^{2k}(\lambda_{22}+\lambda_{21}\xi-\lambda_{12}\xi^*-\lambda_{11}|\xi|^2)^{-2k} \tag{A1.3}$$

where

$$\begin{aligned}\lambda_{11}&=\mathrm{e}^{\mathrm{i}\gamma_0/2}-\gamma_+\gamma_-\,\mathrm{e}^{-\mathrm{i}\gamma_0} \qquad &\lambda_{12}&=\gamma_+\,\mathrm{e}^{-\mathrm{i}\gamma_0/2}\\ \lambda_{21}&=-\gamma_-\,\mathrm{e}^{-\mathrm{i}\gamma_0/2} \qquad &\lambda_{22}&=\mathrm{e}^{-\mathrm{i}\gamma_0/2}.\end{aligned} \tag{A1.4}$$

Then the Q-representatives may be found for any operator functions of K_+, K_- and K_0 from taking the appropriate derivatives of equation (A1.3). For example, the Q-representatives of K_0 and K_+ are just

$$\langle\xi|K_0|\xi\rangle=\mathrm{i}\frac{\partial\langle\xi|\Lambda|\xi\rangle}{\partial\gamma_0}\bigg|_{\substack{\gamma_0=0\\ \gamma_+=0\\ \gamma_-=0}}=k\left(\frac{1+|\xi|^2}{1-|\xi|^2}\right)=k\cosh\theta \tag{A1.5}$$

$$\langle\xi|K_+|\xi\rangle=\frac{\partial\langle\xi|\Lambda|\xi\rangle}{\partial\gamma_+}\bigg|_{\substack{\gamma_0=0\\ \gamma_+=0\\ \gamma_-=0}}=\frac{2k\xi^*}{1-|\xi|^2}=-k\sinh\theta\,\mathrm{e}^{-\mathrm{i}\phi} \tag{A1.6}$$

and, of course, $\langle K_-\rangle=\langle K_+\rangle^*$.

The P-representative of operator A is

$$A=\int d\mu(\xi)|\xi\rangle A_P(\xi)\langle\xi|. \tag{A1.7}$$

Thus we have the convolution

$$A_Q(\xi')=\langle\xi'|A|\xi'\rangle=\int d\mu(\xi)\ A_P(\xi)|\langle\xi'|\xi\rangle|^2. \tag{A1.8}$$

However, to find a more direct relation between A_Q and A_P let us take the matrix element of A with respect to the basis of $\mathscr{D}^+(k)$ to obtain from equation (A1.7) [29]

$$A_{m',m}=\langle m'k|A|m,k\rangle=\int d\mu(\xi)\ A_P(\xi)\langle m'k|\xi\rangle\langle\xi|m,k\rangle \tag{A1.9}$$

where from equation (2.7) and that $\xi=-\tanh(\theta/2)\,e^{-i\phi}$

$$\langle m,k|\xi\rangle=\left(\frac{\Gamma(m+2k)}{m!\Gamma(2k)}\right)^{1/2} e^{im\phi}(-1)^m\left(\sinh\frac{\theta}{2}\right)^m\left(\cosh\frac{\theta}{2}\right)^{-2k-m}. \tag{A1.10}$$

Let us assume a general form for $A_P(\xi)$,

$$A_P(\xi)=A_P(\theta,\phi)=\sum_{Apq} a_{mpq}\,e^{in\phi}\left(\sinh\frac{\theta}{2}\right)^p\left(\cosh\frac{\theta}{2}\right)^q. \tag{A1.11}$$

Then with the transformation to the variables θ, ϕ

$$\frac{d^2\xi}{(1-|\xi|^2)^2}=\frac{1}{2}\sinh\frac{\theta}{2}\cosh\frac{\theta}{2}\,d\theta\,d\phi \tag{A1.12}$$

we have

$$\begin{aligned}A_{m'm}=\frac{2k-1}{2\pi}(-1)^{m+m'}&\left(\frac{\Gamma(m+2k)\Gamma(m'+2k)}{m!m'!\Gamma^2(2k)}\right)^{1/2}\\ &\times\int_0^{2\pi}d\phi\int_0^\infty d\theta\sum_{npq} a_{npq}\,e^{i(n-m'+m)\phi}\left(\sinh\frac{\theta}{2}\right)^{m+m'+p+1}\\ &\times\left(\cosh\frac{\theta}{2}\right)^{-4k-m-m'+q+1}.\end{aligned} \tag{A1.13}$$

The integration over the ϕ-variable contributes the factors $\delta_{n,m'-m}$. To calculate the θ integrals make the change of variable $u=\theta/2$ and use

$$\int_0^\infty du(\sinh u)^\mu(\cosh u)^{-\nu}=\frac{1}{2}B\left(\frac{\mu+\nu}{2},\frac{\nu-\mu}{2}\right) \tag{A1.14}$$

which is valid for $\mathrm{Re}(\nu-\mu)>0$. Thus we obtain

$$\begin{aligned}A_{m',m}=(-1)^{m+m'}[\Gamma(2k-1)]^{-1}&\left(\frac{\Gamma(m+2k)\Gamma(m'+2k)}{\Gamma(m+1)\Gamma(m'+1)}\right)^{1/2}\\ &\times\sum_{npq} a_{npq}\delta_{n,m'-m}B\left(\frac{m+m'+p+2}{2},\frac{4k-p-q-2}{2}\right).\end{aligned} \tag{A1.15}$$

Thus, knowing the matrix elements $A_{m',m}$, we may choose the coefficients a_{npq} to make equation (A1.15) an equality.

For example, for the operator K_+, the matrix elements are $(K_+)_{m'm} = [(m+1)(m+2k)]^{1/2}\delta_{m',m+1}$. This implies that the only non-zero coefficient in equation (A1.15) is a_{110} such that $a_{110} = -2(k-1)$. The P-representative of K_+ is therefore

$$K_{+P}(\theta, \phi) = -2(k-1)\,\mathrm{e}^{-\mathrm{i}\phi} \sinh\left(\frac{\theta}{2}\right) \cosh\left(\frac{\theta}{2}\right) = -(k-1)\,\mathrm{e}^{-\mathrm{i}\phi} \sinh\theta \tag{A1.16}$$

or in terms of ξ

$$K_{+P}(\xi) = \frac{2(k-1)\xi^*}{(1-|\xi|^2)}. \tag{A1.17}$$

For the operator K_0, the matrix elements are $(K_0)_{m',m} = (m+k)\delta_{m',m}$, which implies that only a_{002} and a_{020} are non-zero such that $a_{002} = a_{020} = k-1$. Thus the P-representative of K_0 is

$$K_{0P}(\theta, \phi) = (k-1)\left[\sinh^2\left(\frac{\theta}{2}\right) + \cosh^2\left(\frac{\theta}{2}\right)\right] = (k-1)\cosh\theta \tag{A1.18}$$

or

$$K_{0P}(\xi) = (k-1)\left(\frac{1+|\xi|^2}{1-|\xi|^2}\right). \tag{A1.19}$$

Appendix 2

In this appendix we derive the differential equation satisfied by the quantum Q-function in the case of the anharmonic oscillator. Taking the expectation value of equation (4.18) we apparently must evaluate $\langle\xi|\rho K_+K_-|\xi\rangle$ and $\langle\xi|K_+K_-\rho|\xi\rangle$ where $Q(\xi, t) = \langle\xi|\rho(t)|\xi\rangle$. It is sufficient to evaluate only the former.

We start with the following identity:

$$K_- = 2\xi K_0 - \xi^2 K_+ + \mathrm{e}^{\xi K_+} K_-\,\mathrm{e}^{-\xi K_+}. \tag{A2.1}$$

Since

$$\mathrm{e}^{-\xi K_+}\,\mathrm{e}^{\xi K_+}|0, k\rangle = |0, k\rangle \tag{A2.2}$$

and $K_-|0, k\rangle = 0$ then

$$K_-|\xi\rangle = (2\xi K_0 - \xi^2 K_+)|\xi\rangle. \tag{A2.3}$$

Thus

$$\langle\xi|\rho K_+K_-|\xi\rangle = 2\xi\langle\xi|\rho K_+K_0|\xi\rangle - \xi^2\langle\xi|\rho K_+^2|\xi\rangle. \tag{A2.4}$$

Using the identity

$$K_0 = \xi K_+ + \mathrm{e}^{\xi K_+} K_0\,\mathrm{e}^{-\xi K_+} \tag{A2.5}$$

then

$$K_0|\xi\rangle = \xi K_+|\xi\rangle + \mathrm{e}^{\xi K_+} K_0|0, k\rangle(1-|\xi|^2)^k. \tag{A2.6}$$

But since $K_0|0, k\rangle = k|0, k\rangle$ we have

$$K_0|\xi\rangle = \xi K_+|\xi\rangle + k|\xi\rangle. \tag{A2.7}$$

Thus

$$\langle\xi|\rho K_+K_0|\xi\rangle = \xi\langle\xi|\rho K_+^2|\xi\rangle + k\langle\xi|\rho K_+|\xi\rangle \tag{A2.8}$$

and therefore

$$\langle\xi|\rho K_+K_-|\xi\rangle = \xi^2\langle\xi|\rho K_+^2|\xi\rangle + 2k\xi\langle\xi|\rho K_+|\xi\rangle. \tag{A2.9}$$

Now since Q is defined as

$$Q = \langle\xi|\rho|\xi\rangle = (1-|\xi|^2)^{2k}\langle 0, k|\,e^{\xi^* K_-}\rho\, e^{\xi K_+}|0, k\rangle \tag{A2.10}$$

then we have

$$\frac{\partial Q}{\partial\xi} = -\frac{2k\xi^*}{(1-|\xi|^2)}\,Q + \langle\xi|\rho K_+|\xi\rangle \tag{A2.11}$$

or

$$\langle\xi|\rho K_+|\xi\rangle = \frac{2k\xi^*}{(1-|\xi|^2)}\,Q + \frac{\partial Q}{\partial\xi}. \tag{A2.12}$$

In a like manner we obtain

$$\langle\xi|\rho K_+^2|\xi\rangle = \frac{\partial^2 Q}{\partial\xi^2} + \frac{4k\xi^*}{(1-|\xi|^2)}\frac{\partial Q}{\partial\xi} + \frac{(4k^2-2k)}{(1-|\xi|^2)^2}\,\xi^{*2}Q. \tag{A2.13}$$

Thus we have

$$\langle\xi|\rho K_+K_-|\xi\rangle = \xi^2\frac{\partial^2 Q}{\partial\xi^2} + 2k\left(\frac{1+|\xi|^2}{1-|\xi|^2}\right)\xi\frac{\partial Q}{\partial\xi} + \left(\frac{4k^2|\xi|^2 - 2k|\xi|^4}{(1-|\xi|^2)^2}\right)Q. \tag{A2.14}$$

Since $K_+K_-\rho = (\rho K_+K_-)^+$ the $\langle\xi|K_+K_-\rho|\xi\rangle$ is just the complex conjugate of equation (A2.14).

In the use of the degenerate parametric amplifier we need $\langle\xi|K_-\rho|\xi\rangle$, which is the complex conjugate of equation (A2.12). We also need $\langle\xi|\rho K_-|\xi\rangle$ and $\langle\xi|K_+\rho|\xi\rangle$. Using the identity of equation (A2.1) we have

$$\langle\xi|\rho K_-|\xi\rangle = 2\xi\langle\xi|\rho K_0|\xi\rangle - \xi^2\langle\xi|K_+|\xi\rangle. \tag{A2.15}$$

But since $K_0|\xi\rangle = \xi K_+|\xi\rangle + k|\xi\rangle$ then

$$\langle\xi|\rho K_-|\xi\rangle = \xi^2\langle\xi|\rho K_+|\xi\rangle + 2k\xi Q \tag{A2.16}$$

where $\langle\xi|\rho K_+|\xi\rangle$ is given by equation (A2.12).

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